

Non-Autonomous Forms and Invariance

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Abstract

We generalize the Beurling–Deny–Ouhabaz criterion for parabolic evolution equations governed by forms to the non-autonomous, non-homogeneous and semilinear case. Let V, H be Hilbert spaces such that V is continuously and densely embedded in H and let $\mathcal{A}(t): V \rightarrow V'$ be the operator associated with a bounded H -elliptic form $\mathfrak{a}(t, \cdot, \cdot): V \times V \rightarrow \mathbb{C}$ for all $t \in [0, T]$. Suppose $\mathcal{C} \subset H$ is closed and convex and $P: H \rightarrow H$ the orthogonal projection onto \mathcal{C} . Given $f \in L^2(0, T; V')$ and $u_0 \in \mathcal{C}$, we investigate whenever the solution of the non-autonomous evolutionary problem

$$u'(t) + \mathcal{A}(t)u(t) = f(t), \quad u(0) = u_0,$$

remains in \mathcal{C} and show that this is the case if

$$Pu(t) \in V \quad \text{and} \quad \operatorname{Re} \mathfrak{a}(t, Pu(t), u(t) - Pu(t)) \geq \operatorname{Re} \langle f(t), u(t) - Pu(t) \rangle$$

for a.e. $t \in [0, T]$. Moreover, we examine necessity of this condition and apply this result to a semilinear problem.

Key words: Sesquilinear forms, non-autonomous evolution equations, invariance of closed convex sets.

MSC: 35K90, 35K58.

1 Introduction

The aim of the present article is to generalize the Beurling–Deny–Ouhabaz criterion for parabolic evolution equations to the non-autonomous, non-homogeneous and semilinear case. We first consider an non-homogeneous Cauchy problem of the form

$$u'(t) + \mathcal{A}(t)u(t) = f(t) \text{ for a.e. } t \in (0, T), \quad u(0) = u_0. \quad (1.1)$$

on the Hilbert space V' where $V \xrightarrow{d} H$, V' is the antidual of V and the operators $\mathcal{A}(t)$ are associated with a bounded, quasi-coercive non-autonomous form $\mathfrak{a}: [0, T] \times V \times V \rightarrow \mathbb{K}$ (see Section 2 for precise definitions). If $f \in L^2(0, T; V')$ and $u_0 \in H$, then by a classical theorem of Lions, there exists a unique solution $u \in MR([0, T]) := H^1(0, T; V') \cap L^2(0, T; V)$. Note that $V \xrightarrow{d} H$ and so $H \xrightarrow{d} V'$. The maximal regularity space $MR([0, T])$ can be identified with a subspace of $C([0, T]; H)$. Thus the initial condition $u(0) = u_0$ is meaningful.

Let $\mathcal{C} \subset H$ be closed and convex and let $P: H \rightarrow H$ be the orthogonal projection onto \mathcal{C} . Now if \mathfrak{a} is *autonomous*, i.e. $\mathfrak{a}(\cdot, v, w)$ is constant for every $v, w \in V$, and $f = 0$, then the Beurling–Deny–Ouhabaz criterion states the following.

Theorem 1.1 ([MVV05, Theorem 2.1], [Ouh05, Theorem 2.2]). *Every $u \in MR([0, T])$ with $u' + \mathcal{A}u = 0$, $u(0) \in \mathcal{C}$ satisfies $u(t) \in \mathcal{C}$ for every $t \in [0, T]$ if and only if $PV \subset V$ and $\operatorname{Re} \mathfrak{a}(Pv, v - Pv) \geq 0$ for every $v \in V$.*

A first non-autonomous version of the result stated above was given in [ADO14]: Let $f \in L^2(0, T; V')$. Suppose that

$$PV \subset V \quad \text{and} \quad \operatorname{Re} \mathfrak{a}(t, Pv, v - Pv) \geq \operatorname{Re} \langle f(t), v - Pv \rangle \quad (v \in V), \quad (1.2)$$

for a.e. $t \in [0, T]$, then every $u \in MR([0, T])$ with $u' + \mathcal{A}u = f$, $u(0) \in \mathcal{C}$ satisfies $u(t) \in \mathcal{C}$ for every $t \in [0, T]$.

This result is very satisfying for the nonhomogeneous equation, but is not suitable for semilinear equations (see below). For this reason we establish in Section 3 a different criterion. We show that $u \in MR([0, T])$ with $u' + \mathcal{A}u = f$, $u(0) \in \mathcal{C}$ satisfies $u(t) \in \mathcal{C}$ for every $t \in [0, T]$ provided that

$$Pu(t) \in V \quad \text{and} \quad \operatorname{Re} \mathfrak{a}(t, Pu(t), u(t) - Pu(t)) \geq \operatorname{Re} \langle f(t), u(t) - Pu(t) \rangle$$

for a.e. $t \in [0, T]$. Thus we do not have to test for all $v \in V$ but just by $u(t)$ for a.e. $t \in [0, T]$. This weaker condition makes a big difference when semilinear problems are considered (see Corollary 3.3). The crucial point in the proof is a version of the fundamental theorem of calculus involving the form and the projection under the natural regularity hypotheses (Lemma 3.6). Its proof requires completely new ideas.

So far, in the non-autonomous and even in the autonomous non-homogeneous case (i.e. $f \neq 0$), nothing is known on necessity of (1.2). In Section 4 we show under mild regularity assumptions that (1.2) holds if every solution $u \in MR([t_0, b])$ of $u' + \mathcal{A}u = f$, $u(t_0) \in \mathcal{C}$, $t_0 \in [a, b]$ has values in \mathcal{C} (see Corollary 4.2).

Also for the semilinear equation we prove necessity of the condition, however some more refined arguments are needed (see Section 6).

In Section 7 we give an illustrating example of a semilinear equation with logistic growth.

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2 Preliminaries

Let \mathbb{K} be the field \mathbb{R} or \mathbb{C} and let V and H be Hilbert spaces over the field \mathbb{K} such that $V \xhookrightarrow{d} H$; i.e., V is continuously and densely embedded in H . Then $H \xhookrightarrow{d} V'$ via $v \mapsto (v | \cdot)_H$, where V' denotes the antidual (or dual if $\mathbb{K} = \mathbb{R}$) of V . Let $I := [a, b]$, where $-\infty < a < b < \infty$ and let $\mathfrak{a}: I \times V \times V \rightarrow \mathbb{K}$. If $\mathfrak{a}(t, \cdot, \cdot)$

is sesquilinear for all $t \in I$ and $\mathfrak{a}(\cdot, v, w)$ is measurable for all $v, w \in V$, we say \mathfrak{a} is a *non-autonomous form*. If there exist a constant $M \geq 0$ such that

$$|\mathfrak{a}(t, v, w)| \leq M \|v\|_V \|w\|_V \quad (t \in I, v, w \in V), \quad (2.1)$$

we say that \mathfrak{a} is *bounded* and if there exist constants $\alpha > 0$ and $\omega \in \mathbb{R}$ such that

$$\operatorname{Re} \mathfrak{a}(t, v, v) + \omega \|v\|_H^2 \geq \alpha \|v\|_V^2 \quad (t \in I, v \in V). \quad (2.2)$$

we say that \mathfrak{a} is *H-elliptic* and *coercive* in the case where $\omega = 0$. If \mathfrak{a} is a bounded H-elliptic non-autonomous form we write $\mathfrak{a} \in (I; V, H)$.

Let $\mathfrak{a} \in (I; V, H)$. For $t \in I$ we define $\mathcal{A}(t) \in \mathcal{L}(V, V')$ by $v \mapsto \mathfrak{a}(t, v, \cdot)$. Then there exists an operator $\tilde{\mathcal{A}} \in \mathcal{L}(L^2(I; V); L^2(I; V'))$ such that $\tilde{\mathcal{A}}u(t) = \mathcal{A}(t)u(t)$ for a.e. $t \in I$. By an abuse of notation we use the same letter for \mathcal{A} and $\tilde{\mathcal{A}}$ and we say that \mathcal{A} is the *operator associated with* \mathfrak{a} and write $\mathcal{A} \sim \mathfrak{a}$. In the separable case this is quite obvious, but it needs special attention in the non-separable case (see [DZ16, Proposition 4.1]).

We define the *maximal regularity space* $MR(I; V, H) := H^1(I; V') \cap L^2(I; V)$, with norm $\|u\|_{MR(I; V, H)}^2 := \|u'\|_{L^2(I; V')}^2 + \|u\|_{L^2(I; V)}^2$. Note that $MR(I; V, H) \hookrightarrow C(I; H)$, thus we consider $MR(I; V, H)$ as a subspace of $C(I; H)$. Moreover, $MR(I; V, H)$ is a Hilbert space for the norm $\|\cdot\|_{MR(I; V, H)}$. If no confusion occurs, we write $MR(I)$ instead of $MR(I; V, H)$. A famous result due to J. L. Lions (see [DL92, p. 513], [Sho97, p. 106]) establishes existence and uniqueness of the Cauchy problem associated with \mathcal{A} .

Theorem 2.1. *Let $\mathfrak{a} \in (I; V, H)$. Then for every $u_a \in H$ and $f \in L^2(I; V')$ there exists a unique $u \in MR(I)$ such that*

$$u' + \mathcal{A}u = f, \quad u(a) = u_a \quad (2.3)$$

Moreover, there exists a constant c_a depending only on M , α and ω such that

$$\|u\|_{MR(I)}^2 \leq c_a \left[\|f\|_{L^2(I; V')}^2 + \|u_a\|_H^2 \right]. \quad (2.4)$$

Note that u' , $\mathcal{A}u$ and f are in $L^2(I; V')$ so we consider the equality $u' + \mathcal{A}u = f$ in the space $L^2(I; V')$. Thus we have *maximal regularity* in the space $L^2(I; V')$. In the following we call the function $u \in MR(I)$ the *solution* of (2.3).

Lemma 2.2 ([Sho97, p. 106]). *Let $u \in MR(I)$. Then $\|u\|_H^2 \in W^{1,1}(I)$ with $(\|u\|_H^2)' = 2 \operatorname{Re} \langle u', u \rangle$.*

As a consequence we obtain that

$$\|u(t)\|_H^2 - \|u(s)\|_H^2 = 2 \int_s^t \operatorname{Re} \langle u', u \rangle \, dt \quad (s, t \in I),$$

since $MR(I) \hookrightarrow C(I; H)$ and $u \mapsto \|u\|_H^2$ is continuous.

3 Invariance of closed convex sets

Let $I := [a, b]$ where $-\infty < a < b < \infty$ and let V, H be Hilbert spaces over the field \mathbb{K} such that $V \xhookrightarrow{d} H$. Suppose $\mathfrak{a} \in (I; V, H)$ and $\mathcal{A} \sim \mathfrak{a}$. Let $\mathcal{C} \subset H$ be a

closed convex set and let $P: H \rightarrow \mathcal{C}$ be the orthogonal projection onto \mathcal{C} ; i.e., for $x \in H$, Px is the unique element in \mathcal{C} such that

$$\|x - Px\|_H \leq \|x - y\|_H \quad (y \in \mathcal{C}).$$

In this section we study invariance properties of the solution u of (2.3) in terms of the form \mathfrak{a} and the projection P . Our main result in this section is the following invariance criterion. The point is that a criterion on an individual solution is given.

Theorem 3.1. *Let $u_a \in \mathcal{C}$, $f \in L^2(I; V')$ and $u \in MR(I)$ be the solution of $u' + \mathcal{A}u = f$, $u(a) = u_a$. Then $u(t) \in \mathcal{C}$ for all $t \in I$ if*

$$Pu(t) \in V \quad \text{and} \quad \operatorname{Re} \mathfrak{a}(t, Pu(t), u(t) - Pu(t)) \geq \operatorname{Re} \langle f(t), u(t) - Pu(t) \rangle \quad (3.1)$$

for a.e. $t \in I$.

As a corollary we obtain a result obtained in [ADO14, Theorem 2.2] which is a criterion for invariance of all solutions.

Corollary 3.2. *Let $f \in L^2(I; V')$. Suppose that $PV \subset V$ and for a.e. $t \in I$*

$$\operatorname{Re} \mathfrak{a}(t, Pv, v - Pv) \geq \operatorname{Re} \langle f(t), v - Pv \rangle \quad (v \in V). \quad (3.2)$$

Then for every solution $u \in MR(I)$ of $u' + \mathcal{A}u = f$ with $u(a) \in \mathcal{C}$, we have $u(t) \in \mathcal{C}$ for every $t \in I$.

The advantage of Theorem 3.1 in contrast to Corollary 3.2 is that we have to test merely by the solution itself. This becomes particularly important if we consider semilinear problems as the following criterion shows.

Corollary 3.3. *Let $F: I \times \mathcal{C} \rightarrow V'$ be a function. Assume that $PV \subset V$ and*

$$\operatorname{Re} \mathfrak{a}(t, Pv, v - Pv) \geq \operatorname{Re} \langle F(t, Pv), v - Pv \rangle \quad (v \in V),$$

for a.e. $t \in I$. Let $u \in MR(I)$ such that $u'(t) + \mathcal{A}u(t) = F(t, Pu(t))$ for a.e. $t \in I$. If $u(a) \in \mathcal{C}$, then $u(t) \in \mathcal{C}$ for every $t \in I$. Consequently it is a solution of $u'(t) + \mathcal{A}u(t) = F(t, u)$, t -a.e.

Proof. Let $f = u' + \mathcal{A}u$. Then $f \in L^2(I; V')$ and (3.1) is satisfied for a.e. $t \in I$ (but possibly not (3.2)). The claim follows from Theorem 3.1. \square

As indicated in the proof, Corollary 3.3 cannot be deduced from Corollary 3.2. In the remainder of this section we prove the following theorem, which contains the assertion of Theorem 3.1.

Theorem 3.4. *Let $u_a \in H$, $f \in L^2(I; V')$ and $u \in MR(I)$ be the solution of $u' + \mathcal{A}u = f$, $u(a) = u_a$. Suppose that*

$$Pu(t) \in V \quad \text{and} \quad \operatorname{Re} \mathfrak{a}(t, Pu(t), u(t) - Pu(t)) \geq \operatorname{Re} \langle f(t), u(t) - Pu(t) \rangle \quad (3.3)$$

for a.e. $t \in I$. Then

$$\|u(t) - Pu(t)\|_H \leq \|u_a - Pu_a\|_H e^{\omega(t-a)} \quad (t \in I),$$

where $\omega \in \mathbb{R}$ satisfies $\operatorname{Re} \mathfrak{a}(t, v, v) \geq -\omega \|v\|_H^2$ for every $t \in I$ and $v \in V$.

Note that such a constant ω always exists, since \mathbf{a} is H -elliptic. If \mathbf{a} is coercive, then ω can be chosen to be negative. In that case the solution approaches \mathcal{C} exponentially fast. The following lemmas are crucial ingredients for the proof.

Lemma 3.5. *Let $v \in V$ such that $Pv \in V$ and $h \in V'$. Suppose that*

$$\operatorname{Re} \mathbf{a}(t, Pv, v - Pv) \geq \operatorname{Re} \langle h, v - Pv \rangle \quad (t \in I). \quad (3.4)$$

Then

$$\|v - Pv\|_V^2 \leq \frac{1}{\alpha^2} (M^2 \|v\|_V^2 + \|h\|_{V'}^2) + \frac{2\omega}{\alpha} \|v - Pv\|_H^2,$$

where $M, \alpha > 0$ and ω are constants such that (2.1) and (2.2) hold.

Proof. By H -ellipticity and boundedness of \mathbf{a} and (3.4) we have for $t \in I$

$$\begin{aligned} \alpha \|v - Pv\|_V^2 - \omega \|v - Pv\|_H^2 &\leq \operatorname{Re} \mathbf{a}(t, v - Pv, v - Pv) \\ &\leq \operatorname{Re} \mathbf{a}(t, v, v - Pv) - \operatorname{Re} \langle h, v - Pv \rangle \\ &\leq M \|v\|_V \|v - Pv\|_V + \|h\|_{V'} \|v - Pv\|_V \end{aligned}$$

From this and the inequality $xy \leq \frac{\alpha}{2} x^2 + \frac{1}{2\alpha} y^2$, $x, y \in \mathbb{R}$, we obtain that

$$\frac{\alpha}{2} \|v - Pv\|_V^2 \leq \frac{1}{2\alpha} (M^2 \|v\|_V^2 + \|h\|_{V'}^2) + \omega \|v - Pv\|_H^2. \quad \square$$

Lemma 3.6. *Let $u \in MR(I)$, such that $Pu \in L^2(I; V)$. Then for $t, s \in I$ with $s \leq t$ we have*

$$\|u(t) - Pu(t)\|_H^2 - \|u(s) - Pu(s)\|_H^2 = 2 \int_s^t \operatorname{Re} \langle u'(r), u(r) - Pu(r) \rangle \, dr.$$

Proof. Recall that $MR(I) \hookrightarrow C(I; H)$ and consequently $Pu \in C(I; H)$, since $P: H \rightarrow H$ is a contraction. Hence, it suffices to show $\|u - Pu\|_H^2 \in W^{1,1}(I)$ with

$$(\|u - Pu\|_H^2)' = 2 \operatorname{Re} \langle u', u - Pu \rangle.$$

Let $0 < \delta < b - a$ and $h \in (a, b - \delta)$. We set for $t \in (a, b - \delta)$

$$\theta_h(t) := \langle u(t+h) - u(t), u(t+h) - Pu(t) \rangle + \overline{\langle u(t+h) - u(t), u(t) - Pu(t) \rangle}$$

and

$$\begin{aligned} \eta_h(t) &:= \langle u(t+h) - u(t), u(t+h) - Pu(t+h) \rangle \\ &\quad + \overline{\langle u(t+h) - u(t), u(t) - Pu(t+h) \rangle}. \end{aligned}$$

Since $\frac{1}{h} \int_{\cdot}^{\cdot+h} u'(s) \, ds \rightarrow u'$ in $L^2((a, b-\delta); V')$ and $u(\cdot+h) \rightarrow u$, $Pu(\cdot+h) \rightarrow Pu$ in $L^2((a, b-\delta); V)$ as $h \rightarrow 0$, we have

$$\frac{1}{h} \theta_h, \frac{1}{h} \eta_h \rightarrow 2 \operatorname{Re} \langle u', u - Pu \rangle \quad (h \rightarrow 0)$$

in $L^1((a, b-\delta))$. Moreover, since $Pu(t)$ is the best approximation of $u(t)$ in \mathcal{C} and $Pu(t+h)$ is the best approximation of $u(t+h)$ in \mathcal{C} , we have

$$\eta_h(t) = \|u(t+h) - Pu(t+h)\|_H^2 - \|u(t) - Pu(t+h)\|_H^2$$

$$\begin{aligned} &\leq \|u(t+h) - Pu(t+h)\|_H^2 - \|u(t) - Pu(t)\|_H^2 \\ &\leq \|u(t+h) - Pu(t)\|_H^2 - \|u(t) - Pu(t)\|_H^2 = \theta_h(t). \end{aligned}$$

Let $\varphi \in C_c^\infty(a, b)$, $0 < \delta < b - a$ such that $\text{supp } \varphi \subset (a, b - \delta)$ and $h \in (0, \delta)$, then

$$\begin{aligned} &\frac{1}{h} \int_a^b \|u - Pu\|_H^2 [\varphi(t-h) - \varphi] \, dt \\ &= \frac{1}{h} \int_a^b [\|u(t+h) - Pu(t+h)\|_H^2 - \|u - Pu\|_H^2] \varphi \, dt \\ &\leq \frac{1}{h} \int_a^b [\theta_h \mathbf{1}_{\{\varphi \geq 0\}} + \eta_h \mathbf{1}_{\{\varphi < 0\}}] \varphi \, dt \end{aligned}$$

Now taking the limit $h \rightarrow 0$ shows

$$- \int_a^b \|u - Pu\|_H^2 \varphi' \, dt \leq \int_a^b 2 \operatorname{Re} \langle u', u - Pu \rangle \varphi \, dt. \quad (3.5)$$

Finally, if we replace φ by $-\varphi$ we obtain equality in (3.5). \square

Proof of Theorem 3.4. Let $u_a \in H$, $f \in L^2(I; V')$ and $u \in MR(I)$ be the solution of $u' + \mathcal{A}u = f$, $u(a) = u_a$. Suppose that (3.3) holds for a.e. $t \in I$. By Lemma 3.5 and (3.3) we obtain that $Pu \in L^2(I; V)$. Thus, by Lemma 3.6 for all $t \in I$ we have

$$\begin{aligned} &\|u(t) - Pu(t)\|_H^2 - \|u_a - Pu_a\|_H^2 = 2 \int_a^t \operatorname{Re} \langle u', u - Pu \rangle \, ds \\ &= 2 \int_a^t \operatorname{Re} \langle f - \mathcal{A}u, u - Pu \rangle \, ds \leq -2 \int_a^t \operatorname{Re} \mathfrak{a}(s, u - Pu, u - Pu) \, ds \end{aligned}$$

where we used the assumption (3.3) for the inequality. Thus

$$\|u(t) - Pu(t)\|_H^2 \leq \|u_a - Pu_a\|_H^2 + 2\omega \int_a^t \|u - Pu\|_H^2 \, ds \quad (t \in I).$$

Now the claim of the theorem follows by Gronwall's lemma. \square

4 Necessity

Let $I := [a, b]$ where $-\infty < a < b < \infty$ and let V, H be Hilbert spaces over the field \mathbb{K} such that $V \xrightarrow{d} H$. Suppose $\mathfrak{a} \in (I; V, H)$, $\mathcal{A} \sim \mathfrak{a}$ and $f \in L^2(I; V')$. Let $\mathcal{C} \subset H$ be a closed convex set and let $P: H \rightarrow \mathcal{C}$ be the orthogonal projection onto \mathcal{C} .

We say that (\mathfrak{a}, f) is \mathcal{C} invariant if for every $c \in I$ and every $u \in MR([c, b])$ with $u' + \mathcal{A}u = f$, $u(c) \in \mathcal{C}$ we have $u(t) \in \mathcal{C}$ for all $t \in [c, b]$.

Theorem 4.1. *Suppose (\mathfrak{a}, f) is \mathcal{C} invariant. Then for every $u \in MR(I)$ with $u' + \mathcal{A}u = f$ we have $Pu \in L^2(I; V)$ and*

$$\operatorname{Re} \mathfrak{a}(t, Pu(t), u(t) - Pu(t)) \geq \operatorname{Re} \langle f(t), u(t) - Pu(t) \rangle \quad (\text{a.e. } t \in I). \quad (4.1)$$

Note that if $u(a) \in \mathcal{C}$ in the theorem above, then (4.1) holds trivially, since $u(t) - Pu(t) = 0$ for all $t \in [a, b]$. But it is remarkable that (4.1) holds for any initial value $u(a) \in H$.

Next we want to deduce a pointwise version from Theorem 4.1, which is in the spirit of the Beurling–Deny–Ouhabaz criterion. Some regularity assumptions are needed for the proof. We say that \mathfrak{a} is *right-continuous* if $\lim_{t \downarrow c} \|\mathcal{A}(c) - \mathcal{A}(t)\|_{\mathcal{L}(V, V')} = 0$ for every $c \in I$.

Corollary 4.2. *Suppose that \mathfrak{a} is right-continuous and that there exists a dense subspace \tilde{V} of V , such that for every $c \in I$, $u_c \in \tilde{V}$ the solution $u \in MR([c, b])$ of $u' + \mathcal{A}u = f$, $u(c) = u_c$ is in $C([c, b]; V)$. Then (\mathfrak{a}, f) is \mathcal{C} invariant if and only if $PV \subset V$ and*

$$\operatorname{Re} \mathfrak{a}(t, Pv, v - Pv) \geq \operatorname{Re} \langle f(t), v - Pv \rangle \quad (a.e. \ t \in I, v \in V). \quad (4.2)$$

For example, if $f \in L^2(I; H)$ and \mathfrak{a} is symmetric and of bounded variation, then for every $c \in I$ every solution $u \in MR([c, b])$ of $u' + \mathcal{A}u = f$, $u_c \in V$ is in $C([c, b]; V)$ (see [Die15]). Another example is the situation where \mathfrak{a} is autonomous, i.e. $\mathfrak{a}(\cdot, v, w)$ is constant for every $v, w \in V$ and $f = 0$. Then every solution of $u' + \mathcal{A}u = 0$, $u(a) \in D(A)$ is in $C(I; D(A))$, where $D(A) := \{v \in V : \mathcal{A}v \in H\}$, $\|v\|_{D(A)}^2 = \|\mathcal{A}v\|_H^2 + \|v\|_H^2$ is densely embedded in V . Thus we recover the Beurling–Deny–Ouhabaz criterion.

Recall that in Corollary 4.2 condition (4.2) is sufficient even if the additional regularity assumptions are not satisfied. The author does not know whether the other implication is true without these assumptions.

Proof of Corollary 4.2. Let $N_1 \subset I$ be a nullset such that $I \setminus N_1$ are Lebesgue points of f . Let $c \in [a, b] \setminus N_1$, $u_c \in \tilde{V}$ and $u \in MR([c, b])$ be the solution of $u' + \mathcal{A}u = f$, $u(c) = u_c$. By Theorem 4.1 we obtain that $Pu \in L^2([c, b]; V)$ and that there exists a nullset $N_2 \subset [c, b]$ such that

$$\operatorname{Re} \mathfrak{a}(t, Pu(t), u(t) - Pu(t)) \geq \operatorname{Re} \langle f(t), u(t) - Pu(t) \rangle \quad (4.3)$$

for $t \in [c, b] \setminus N_2$. Let $(t_n)_{n \in \mathbb{N}} \subset (c, b] \setminus N_2$ be a sequence such that $t_n \downarrow c$ and $f(t_n) \rightarrow f(c)$ in V' for $n \rightarrow \infty$. Note that such a sequence exists by Lebesgue's differentiation theorem. By Lemma 3.5 we obtain that $(Pu(t_n))_{n \in \mathbb{N}}$ is bounded in V , thus we conclude that $Pu(c) \in V$ and $Pu(t_n) \rightarrow Pu(c)$ in V . Thus

$$\begin{aligned} & \operatorname{Re} \mathfrak{a}(c, u(c) - Pu(c), u(c) - Pu(c)) \\ & \leq \limsup_{n \rightarrow \infty} \operatorname{Re} \mathfrak{a}(t_n, u(t_n) - Pu(t_n), u(t_n) - Pu(t_n)) \\ & \leq \limsup_{n \rightarrow \infty} \operatorname{Re} \langle \mathcal{A}(t_n)u(t_n) - f(t_n), u(t_n) - Pu(t_n) \rangle \\ & = \operatorname{Re} \langle \mathcal{A}(c)u(c) - f(c), u(c) - Pu(c) \rangle, \end{aligned}$$

where we used that \mathfrak{a} is right-continuous and $v \mapsto \operatorname{Re} \mathfrak{a}(c, v, v) + \omega \|v\|_H^2$ is an equivalent norm on V for the first inequality and (4.3) with $t = t_n$ for the second inequality. This shows $P\tilde{V} \subset V$ and

$$\operatorname{Re} \mathfrak{a}(t, Pv, v - Pv) \geq \operatorname{Re} \langle f(t), v - Pv \rangle \quad (t \in [a, b] \setminus N_1, v \in \tilde{V}).$$

Finally let $v \in V$ and $(v_n)_{n \in \mathbb{N}} \subset \tilde{V}$, $v_n \rightarrow v$ in V . With a similar argument as above (where we replace the role of $u(t_n)$ by v_n and $u(c)$ by v) we obtain the assertion of the corollary. \square

We finish this section with the proof of the theorem.

Proof of Theorem 4.1. For $n \in \mathbb{N}$ and $k \in \{0, 1, \dots, n\}$, let $t_k^n := a + \frac{k}{n(b-a)}$. Let $v_{n,k} \in MR([t_{k-1}^n, t_k^n])$ be the solution of $v'_{n,k} + \mathcal{A}v_{n,k} = f$, $v_{n,k}(t_{k-1}^n) = Pu(t_{k-1}^n)$ and $v_n \in L^2(I; V)$, $v_n(t) := v_{n,k}(t)$ for $t \in [t_{k-1}^n, t_k^n]$, $k \in \{1, \dots, n\}$. Let $k \in \{1, \dots, n\}$. Since (\mathbf{a}, f) is \mathcal{C} invariant we have that $v_{n,k}(t) \in \mathcal{C}$ for all $t \in [t_{k-1}^n, t_k^n]$. Thus $\|u(t) - Pu(t)\|_H \leq \|u(t) - v_{n,k}(t)\|_H$ for all $t \in [t_{k-1}^n, t_k^n]$. We set $\tilde{u} := u - Pu$, $\tilde{v}_{n,k} := u - v_{n,k}$ and $\tilde{v}_n := u - v_n$ and obtain by Lemma 2.2

$$\begin{aligned} \|\tilde{u}(b)\|_H^2 - \|\tilde{u}(a)\|_H^2 &= \sum_{k=1}^n (\|\tilde{u}(t_k^n)\|_H^2 - \|\tilde{u}(t_{k-1}^n)\|_H^2) \\ &\leq \sum_{k=1}^n (\|\tilde{v}_{n,k}(t_k^n)\|_H^2 - \|\tilde{v}_{n,k}(t_{k-1}^n)\|_H^2) = \sum_{k=1}^n 2 \operatorname{Re} \int_{t_{k-1}^n}^{t_k^n} \langle \tilde{v}'_{n,k}, \tilde{v}_{n,k} \rangle \, ds \\ &= - \int_a^b 2 \operatorname{Re} \mathbf{a}(s, \tilde{v}_n, \tilde{v}_n) \, ds. \end{aligned} \quad (4.4)$$

Suppose at first that $\tilde{v}_n \rightarrow \tilde{u}$ in $L^2(I; H)$. From (4.4) and H -ellipticity of \mathbf{a} we deduce that $\tilde{u} \in L^2(I; V)$, $\tilde{v}_n \rightharpoonup \tilde{u}$ in $L^2(I; V)$ and

$$\|\tilde{u}(b)\|_H^2 - \|\tilde{u}(a)\|_H^2 \leq - \int_a^b 2 \operatorname{Re} \mathbf{a}(s, \tilde{u}, \tilde{u}) \, ds. \quad (4.5)$$

By (4.5) and Lemma 3.6 we obtain

$$\begin{aligned} - \int_a^b 2 \operatorname{Re} \mathbf{a}(s, \tilde{u}, \tilde{u}) \, ds &\geq \|\tilde{u}(b)\|_H^2 - \|\tilde{u}(a)\|_H^2 \\ &= 2 \int_a^b \operatorname{Re} \langle u', \tilde{u} \rangle \, ds = 2 \int_a^b \operatorname{Re} \langle f - \mathcal{A}u, \tilde{u} \rangle \, ds. \end{aligned}$$

Hence

$$0 \geq \int_a^b \operatorname{Re} \langle f - \mathcal{A}Pu, \tilde{u} \rangle \, ds.$$

Note that this inequality holds also if we integrate over any interval $J \subset I$ instead of I with a simple modification of the argument above. Applying Lebesgue's differentiation Theorem this finishes the proof if $\tilde{v}_n \rightarrow \tilde{u}$ in $L^2(I; H)$. We have

$$\begin{aligned} &\int_{t_{k-1}^n}^{t_k^n} \|\tilde{v}_n - \tilde{u}\|_H^2 \, ds \\ &\leq 3 \int_{t_{k-1}^n}^{t_k^n} \|\tilde{v}_{n,k} - \tilde{v}_{n,k}(t_{k-1}^n)\|_H^2 \, ds + 6 \int_{t_{k-1}^n}^{t_k^n} \|u(t_{k-1}^n) - u\|_H^2 \, ds \\ &\leq \frac{3C}{n(b-a)} \left(\|\tilde{v}_{n,k}(t_{k-1}^n)\|_H^2 - \|\tilde{v}_{n,k}(t_k^n)\|_H^2 + \|\tilde{v}_{n,k}\|_{L^2(t_{k-1}^n, t_k^n; H)}^2 \right) \\ &+ \frac{6C}{n(b-a)} \left(\|u(t_{k-1}^n)\|_H^2 - \|u(t_k^n)\|_H^2 + \|u\|_{L^2(t_{k-1}^n, t_k^n; H)}^2 + \|f\|_{L^2(t_{k-1}^n, t_k^n; V')}^2 \right) \end{aligned}$$

where we use that $Pu(t_{k-1}^n) = v_{n,k}(t_{k-1}^n)$ and that P is a contraction in the first estimate and Lemma 4.3 below in the second estimate. We take the sum over k from 1 to n and obtain by the first estimate of (4.4)

$$\begin{aligned} \int_a^b \|\tilde{v}_n - \tilde{u}\|_H^2 ds &\leq \frac{3C}{n(b-a)} \left(\|\tilde{u}(a)\|_H^2 - \|\tilde{u}(b)\|_H^2 + \|\tilde{v}_n\|_{L^2(I;H)}^2 \right. \\ &\quad \left. + 2\|u(a)\|_H^2 - 2\|u(b)\|_H^2 + 2\|u\|_{L^2(I;H)}^2 + 2\|f\|_{L^2(I;V')}^2 \right). \end{aligned}$$

By the reverse triangle inequality it follows that $\|\tilde{v}_n\|_{L^2(I;H)}^2$ is bounded. Thus $\tilde{v}_n \rightarrow \tilde{u}$ in $L^2(I;H)$. \square

Lemma 4.3. *Let $u \in MR(I)$. Then there exists a constant $C > 0$ such that*

$$\|u(t) - u(a)\|_H^2 \leq C \left(\|u(a)\|_H^2 - \|u(b)\|_H^2 + \|u\|_{L^2(I;H)}^2 + \|f\|_{L^2(I;V')}^2 \right)$$

for all $t \in I$, where $f := u' + \mathcal{A}u$.

Proof. Let $t \in (a, b]$. We set $v(s) := u(\frac{1}{2}(t+s)) - u(a + \frac{1}{2}(t-s))$. Then $v(a) = 0$, $v(t) = u(t) - u(a)$ and $v \in MR([a, t])$. Thus

$$\begin{aligned} \|u(t) - u(a)\|_H^2 &= \|v(t)\|_H^2 - \|v(a)\|_H^2 = 2 \int_a^t \operatorname{Re}\langle v', v \rangle ds \\ &\leq \|v\|_{MR([a, t])}^2 \leq 2\|u\|_{MR([a, t])}^2 \leq 2\|u\|_{MR(I)}^2. \end{aligned}$$

Moreover,

$$\begin{aligned} \|u(b)\|_H^2 - \|u(a)\|_H^2 &= 2 \int_a^b \operatorname{Re}\langle u', u \rangle ds = 2 \int_a^b \operatorname{Re}\langle f - \mathcal{A}u, u \rangle ds \\ &\leq -2\alpha\|u\|_{L^2(I;V)}^2 + 2\omega\|u\|_{L^2(I;H)}^2 + 2\|f\|_{L^2(I;V')}^2 + 2\|u\|_{L^2(I;V)}^2 \\ &\leq -\alpha\|u\|_{L^2(I;V)}^2 + \omega\|u\|_{L^2(I;H)}^2 + \frac{1}{\alpha}\|f\|_{L^2(I;V')}^2 \end{aligned}$$

and

$$\|u'\|_{L^2(I;V')} = \|f - \mathcal{A}u\|_{L^2(I;V')} \leq \|f\|_{L^2(I;V')} + M\|u\|_{L^2(I;V)}.$$

Now the claim follows by the three estimates above. \square

5 A semilinear problem

In Section 6 we want to study a semilinear version of the necessity conditions for invariance given in Section 4. Before that we want to establish well-posedness at least in a simple case.

Let $I := [a, b]$ where $-\infty < a < b < \infty$ and let V, H be Hilbert spaces over the field \mathbb{K} such that $V \xrightarrow{d} H$. Let $\mathfrak{a} \in (I; V, H)$, $\mathcal{A} \sim \mathfrak{a}$. Suppose $F: I \times H \rightarrow V'$ satisfies $F(\cdot, v) \in L^2(I; V')$ for every $v \in H$ and there exists a constant $L > 0$ such that

$$\|F(t, v) - F(t, w)\|_{V'} \leq L\|v - w\|_H \quad (t \in I, v, w \in H).$$

Proposition 5.1. *For every $u_a \in H$ there exists a unique $u \in MR(I)$ such that $u' + \mathcal{A}u = F(\cdot, u)$, $u(a) = u_a$.*

Before we prove Proposition 5.1 we need several lemmas. We denote by $M \geq 0$, $\alpha > 0$ and $\omega \in \mathbb{R}$ the continuity and ellipticity constants in (2.1) and (2.2).

Lemma 5.2. *Let $u \in MR(I)$ with $u(a) = 0$. Then*

$$\|u\|_{L^2(I;H)}^2 \leq \frac{b-a}{2\sqrt{2}} \|u\|_{MR(I)}^2.$$

Proof. Since $u(a) = 0$ we have

$$\begin{aligned} \|u\|_{L^2(I;V')}^2 &= \int_a^b \left\| \int_a^t u'(s) \, ds \right\|_{V'}^2 dt \leq \int_a^b \int_a^t (t-a) \|u'(s)\|_{V'}^2 \, ds \, dt \\ &\leq \frac{(b-a)^2}{2} \int_a^b \|u'(s)\|_{V'}^2 \, ds = \frac{(b-a)^2}{2} \|u'\|_{L^2(I;V')}^2. \end{aligned}$$

Thus

$$\begin{aligned} \|u\|_{L^2(I;H)}^2 &\leq \|u\|_{L^2(I;V')} \|u\|_{L^2(I;V)} \\ &\leq \frac{b-a}{\sqrt{2}} \|u'\|_{L^2(I;V')} \|u\|_{L^2(I;V)} \leq \frac{b-a}{2\sqrt{2}} \|u\|_{MR(I)}^2. \quad \square \end{aligned}$$

Lemma 5.3. *Let $-\infty < a < c < b < \infty$. Let $u_1 \in MR([a, c])$, $u_2 \in MR([c, b])$ such that $u_1(c) = u_2(c)$. Set $u(t) := u_1(t)$ for $t \in [a, c]$ and $u(t) := u_2(t)$ for $t \in (c, b]$. Then $u \in MR([a, b])$.*

Proof of Proposition 5.1. a) By Theorem 2.1 there exists a constant c_a such that (2.4) holds. Let $b > a$ such that $(b-a) < q := \frac{2\sqrt{2}}{c_a L^2}$ and let $u_a \in H$. We define $S: MR([a, b]) \rightarrow MR([a, b])$, $v \mapsto u$, where u is the solution of $u' + \mathcal{A}u = F(\cdot, v(\cdot))$, $u(a) = u_a$. For $u, v \in MR([a, b])$ we have by Lemma 5.2

$$\begin{aligned} \|Su - Sv\|_{MR([a, b])}^2 &\leq c_a \|F \circ u - F \circ v\|_{L^2(a, b; V')}^2 \\ &\leq c_a L^2 \|u - v\|_{L^2(a, b; H)}^2 \leq c_a L^2 \frac{b-a}{2\sqrt{2}} \|u - v\|_{MR([a, b])}^2. \end{aligned}$$

Thus S is a strict contraction and by the Banach fixed-point theorem we obtain a unique $u \in MR([a, b])$ such that $Su = u$, i.e. u is the unique solution of $u' + \mathcal{A}u = F(\cdot, u(\cdot))$, $u(a) = u_a$.

b) Part a) together with Lemma 5.3 yields a solution $u \in MR([a, b])$ for any $b > a$.

c) We show uniqueness. Let $u_1, u_2 \in MR(I)$ be solutions with $u_1(a) = u_2(a)$. Recall that $u_1, u_2 \in C(I; H)$. Assume that u_1 and u_2 are different, then there exists $t_0 \in [a, b)$ such that $u_1 = u_2$ on $[a, t_0]$ but $u_1(t_n) \neq u_2(t_n)$ for some $t_n \downarrow t_0$. Choose $0 < \varepsilon < \min\{q, T - t_0\}$. Then there exist two different solutions on $[t_0, t_0 + \varepsilon]$ which contradicts a). \square

6 Semilinear Necessity

In Section 3 we saw that the invariance criterion for the non-homogeneous equation (Theorem 3.1) could be applied immediately to semilinear problems. The necessity result (Theorem 4.1) cannot so easily be carried over. Additional arguments are needed to adapt the proofs of Section 4 to the semilinear case.

Let $I := [a, b]$ where $-\infty < a < b < \infty$ and let V, H be Hilbert spaces over the field \mathbb{K} such that $V \xhookrightarrow{d} H$. Let $\mathfrak{a} \in (I; V, H)$, $\mathcal{A} \sim \mathfrak{a}$. Let $F: I \times H \rightarrow V'$ be a function such that $F(\cdot, v) \in L^2(I; V')$ for every $v \in H$ and suppose that there exists a constant $L > 0$ such that

$$\|F(t, v) - F(t, w)\|_{V'} \leq L\|v - w\|_H \quad (t \in I, v, w \in H). \quad (6.1)$$

Then by Proposition 5.1, for every $c \in [a, b]$ and every $u_c \in H$ there exists a unique $u \in MR([c, b])$ such that $u' + \mathcal{A}u = F(\cdot, u)$, $u(c) = u_c$.

Let $\mathcal{C} \subset H$ be a closed convex set and let $P: H \rightarrow \mathcal{C}$ be the orthogonal projection onto \mathcal{C} . We say that (\mathfrak{a}, F) is \mathcal{C} invariant if for every $c \in I$ and every $u \in MR([c, b])$ with $u' + \mathcal{A}u = F(\cdot, u)$, $u(c) \in \mathcal{C}$ we have $u(t) \in \mathcal{C}$ for all $t \in [c, b]$.

Theorem 6.1. *Suppose (\mathfrak{a}, F) is \mathcal{C} invariant. Then for every $u \in MR(I)$ with $u' + \mathcal{A}u = F(\cdot, Pu)$ we have $Pu \in L^2(I; V)$ and*

$$\operatorname{Re} \mathfrak{a}(t, Pu(t), u(t) - Pu(t)) \geq \operatorname{Re} \langle F(t, Pu), u(t) - Pu(t) \rangle \quad (a.e. t \in I).$$

Proof. For $n \in \mathbb{N}$ and $k \in \{0, 1, \dots, n\}$, let $t_k^n := a + \frac{k}{n(b-a)}$. Let $v_{n,k} \in MR([t_{k-1}^n, t_k^n])$ be the solution of $v'_{n,k} + \mathcal{A}v_{n,k} = F(\cdot, v_{n,k})$, $v_{n,k}(t_{k-1}^n) = Pu(t_{k-1}^n)$ and $v_n \in L^2(I; V)$, $v_n(t) := v_{n,k}(t)$ for $t \in [t_{k-1}^n, t_k^n]$, $k \in \{1, \dots, n\}$. Let $k \in \{1, \dots, n\}$. Since (\mathfrak{a}, F) is \mathcal{C} invariant we have that $v_{n,k}(t) \in \mathcal{C}$ for all $t \in [t_{k-1}^n, t_k^n]$. Thus $\|u(t) - Pu(t)\|_H \leq \|u(t) - v_{n,k}(t)\|_H$ for all $t \in [t_{k-1}^n, t_k^n]$. Let $\varepsilon \in (0, 2)$. We set $\tilde{u} := u - Pu$, $\tilde{v}_{n,k} := u - v_{n,k}$ and $\tilde{v}_n := u - v_n$ and obtain

$$\begin{aligned} \|\tilde{u}(b)\|_H^2 - \|\tilde{u}(a)\|_H^2 &= \sum_{k=1}^n (\|\tilde{u}(t_k^n)\|_H^2 - \|\tilde{u}(t_{k-1}^n)\|_H^2) \\ &\leq \sum_{k=1}^n (\|\tilde{v}_{n,k}(t_k^n)\|_H^2 - \|\tilde{v}_{n,k}(t_{k-1}^n)\|_H^2) = \sum_{k=1}^n 2 \operatorname{Re} \int_{t_{k-1}^n}^{t_k^n} \langle \tilde{v}'_{n,k}, \tilde{v}_{n,k} \rangle \, ds \\ &= \int_a^b 2 \operatorname{Re} \langle F(\cdot, u) - F(\cdot, v_n), \tilde{v}_n \rangle \, ds - \int_a^b 2 \operatorname{Re} \mathfrak{a}(s, \tilde{v}_n, \tilde{v}_n) \, ds \\ &\leq 2L \int_a^b \|\tilde{v}_n\|_H \|\tilde{v}_n\|_V \, ds - \int_a^b 2 \operatorname{Re} \mathfrak{a}(s, \tilde{v}_n, \tilde{v}_n) \, ds \\ &\leq 2L \int_a^b \|\tilde{v}_n\|_H \|\tilde{v}_n\|_V \, ds - \varepsilon \int_a^b (\alpha \|\tilde{v}_n\|_V^2 - \omega \|\tilde{v}_n\|_H^2) \, ds \\ &\quad - \int_a^b (2 - \varepsilon) \operatorname{Re} \mathfrak{a}(s, \tilde{v}_n, \tilde{v}_n) \, ds \\ &\leq \int_a^b \left(\varepsilon \omega + \frac{L^2}{\varepsilon \alpha} \right) \|\tilde{v}_n\|_H^2 \, ds - \int_a^b (2 - \varepsilon) \operatorname{Re} \mathfrak{a}(s, \tilde{v}_n, \tilde{v}_n) \, ds \quad (6.2) \end{aligned}$$

where we used Lemma 2.2 in the equality in the second line, (6.1) and that P is a contraction for the second inequality, H -ellipticity of \mathfrak{a} in the third inequality

and Young's inequality ($2xy \leq \varepsilon \alpha x^2 + \frac{1}{\varepsilon \alpha} y^2$, $x, y \in \mathbb{R}$) in the last inequality. Suppose that $\tilde{v}_n \rightarrow \tilde{u}$ in $L^2(I; H)$, then (6.2) yields $\tilde{u} \in L^2(I; V)$ with $\tilde{v}_n \rightharpoonup \tilde{u}$ in $L^2(I; V)$ and

$$\|\tilde{u}(b)\|_H^2 - \|\tilde{u}(a)\|_H^2 \leq - \int_a^b (2 - \varepsilon) \operatorname{Re} \mathfrak{a}(s, \tilde{u}, \tilde{u}) \, ds.$$

Since $\varepsilon \in (0, 2)$ was arbitrary we let $\varepsilon \downarrow 0$ and obtain

$$\|\tilde{u}(b)\|_H^2 - \|\tilde{u}(a)\|_H^2 \leq - \int_a^b 2 \operatorname{Re} \mathfrak{a}(s, \tilde{u}, \tilde{u}) \, ds. \quad (6.3)$$

We combine (6.3) and Lemma 3.6, thus

$$\begin{aligned} - \int_a^b 2 \operatorname{Re} \mathfrak{a}(s, \tilde{u}, \tilde{u}) \, ds &\geq \|\tilde{u}(b)\|_H^2 - \|\tilde{u}(a)\|_H^2 \\ &= 2 \int_a^b \operatorname{Re} \langle u', \tilde{u} \rangle \, ds = 2 \int_a^b \operatorname{Re} \langle F(\cdot, Pu) - \mathcal{A}u, \tilde{u} \rangle \, ds. \end{aligned}$$

Hence

$$0 \geq \int_a^b \operatorname{Re} \langle F(\cdot, Pu) - \mathcal{A}Pu, \tilde{u} \rangle \, ds.$$

Note that this inequality holds also if we integrate over any interval $J \subset I$ instead of I with a simple modification of the argument above. This finishes the proof if $\tilde{v}_n \rightarrow \tilde{u}$ in $L^2(I; H)$. We have

$$\begin{aligned} &\int_{t_{k-1}^n}^{t_k^n} \|\tilde{v}_n - \tilde{u}\|_H^2 \, ds \\ &\leq 3 \int_{t_{k-1}^n}^{t_k^n} \|\tilde{v}_{n,k} - \tilde{v}_{n,k}(t_{k-1}^n)\|_H^2 \, ds + 6 \int_{t_{k-1}^n}^{t_k^n} \|u(t_{k-1}^n) - u\|_H^2 \, ds \\ &\leq \frac{3C}{n(b-a)} \left(\|\tilde{v}_{n,k}(t_{k-1}^n)\|_H^2 - \|\tilde{v}_{n,k}(t_k^n)\|_H^2 + \|\tilde{v}_{n,k}\|_{L^2(t_{k-1}^n, t_k^n; H)}^2 \right. \\ &\quad \left. + \|F(\cdot, Pu) - F(\cdot, v_{n,k})\|_{L^2(t_{k-1}^n, t_k^n; V')}^2 \right) \\ &\quad + \frac{6C}{n(b-a)} \left(\|u(t_{k-1}^n)\|_H^2 - \|u(t_k^n)\|_H^2 \right. \\ &\quad \left. + \|u\|_{L^2(t_{k-1}^n, t_k^n; H)}^2 + \|F(\cdot, Pu)\|_{L^2(t_{k-1}^n, t_k^n; V')}^2 \right), \end{aligned}$$

where we use that $Pu(t_{k-1}^n) = v_{n,k}(t_{k-1}^n)$ and that P is a contraction in the first estimate and Lemma 4.3 in the second estimate. We take the sum over k from 1 to n and obtain by the first estimate of (4.4) and by (6.1) and the contractivity of P

$$\begin{aligned} \int_a^b \|\tilde{v}_n - \tilde{u}\|_H^2 \, ds &\leq \frac{3C}{n(b-a)} \left(\|\tilde{u}(a)\|_H^2 - \|\tilde{u}(b)\|_H^2 + (1+L)\|\tilde{v}_n\|_{L^2(I; H)}^2 \right. \\ &\quad \left. + 2\|u(a)\|_H^2 - 2\|u(b)\|_H^2 + 2\|u\|_{L^2(I; H)}^2 + 2\|F(\cdot, Pu)\|_{L^2(I; V')}^2 \right). \end{aligned}$$

By the reverse triangle inequality it follows that $\|\tilde{v}_n\|_{L^2(I; H)}^2$ is bounded. Thus $\tilde{v}_n \rightarrow \tilde{u}$ in $L^2(I; H)$. \square

Next we also want to deduce a pointwise version in the semilinear setting, which is in the spirit of the Beurling–Deny–Ouhabaz criterion. Again we use regularity assumptions as in Section 4.

Corollary 6.2. *Suppose that \mathfrak{a} and $F(\cdot, v)$, $v \in V$ are right-continuous and there exists a dense subspace \tilde{V} of V , such that for every $c \in I$, $u_c \in \tilde{V}$ the solution $u \in MR([c, b])$ of $u' + \mathcal{A}u = F(\cdot, Pu)$, $u(c) = u_c$ is in $C([c, b]; V)$. Then (\mathfrak{a}, F) is \mathcal{C} invariant if and only if $PV \subset V$ and*

$$\operatorname{Re} \mathfrak{a}(t, Pv, v - Pv) \geq \operatorname{Re} \langle F(t, Pv), v - Pv \rangle \quad (t \in I, v \in V).$$

For example, if $F: I \times H \rightarrow H$ satisfies $F(\cdot, v) \in L^2(I; H)$ for every $v \in H$ and there exists a constant $L > 0$ such that

$$\|F(t, v) - F(t, w)\|_H \leq L\|v - w\|_H \quad (t \in I, v, w \in H).$$

and \mathfrak{a} is of bounded variation and symmetric (see [Die15]) or \mathfrak{a} is Lipschitz continuous and satisfies $D(A^{1/2}) = V$, where A is the part of $\mathcal{A}(0)$ in H (see [ADLO14]), then the assumptions of the corollary above are satisfied.

Before we prove the corollary we state a simple, autonomous version of it. The assumption $D(A^{1/2}) = V$ is called *Kato's square root property*. For example by [AT03] it is satisfied for elliptic operators in divergence form on Lipschitz domains with Dirichlet or Neumann boundary condition.

Corollary 6.3. *Let $F: H \rightarrow H$ be Lipschitz continuous. Suppose that \mathfrak{a} is autonomous, i.e. $\mathfrak{a}(\cdot, v, w)$ is constant for all $v, w \in V$, and that $D(A^{1/2}) = V$. Then (\mathfrak{a}, F) is \mathcal{C} invariant if and only if*

$$\operatorname{Re} \mathfrak{a}(Pv, v - Pv) \geq \operatorname{Re} \langle F(Pv), v - Pv \rangle \quad (v \in V).$$

Proof. Let $c \in [a, b]$, $u_c \in \tilde{V}$ and $u \in MR([c, b])$ be the solution of $u' + \mathcal{A}u = F(\cdot, u)$, $u(c) = u_c$. By Theorem 4.1 we obtain that $Pu \in L^2([c, b]; V)$ and that there exists a nullset $N \subset [c, b]$ such that

$$\operatorname{Re} \mathfrak{a}(t, Pu(t), u(t) - Pu(t)) \geq \operatorname{Re} \langle F(t, Pu(t)), u(t) - Pu(t) \rangle \quad (6.4)$$

for $t \in [c, b] \setminus N$. Let $(t_n)_{n \in \mathbb{N}} \subset (c, b] \setminus N$ be a sequence such that $t_n \downarrow c$, then $F(t_n, u(t_n)) \rightarrow F(c, u(c))$ in V' for $n \rightarrow \infty$. By Lemma 3.5 we obtain that $(Pu(t_n))_{n \in \mathbb{N}}$ is bounded in V , thus we conclude that $Pu(c) \in V$ and $Pu(t_n) \rightarrow Pu(c)$ in V . Thus

$$\begin{aligned} & \operatorname{Re} \mathfrak{a}(c, u(c) - Pu(c), u(c) - Pu(c)) \\ & \leq \limsup_{n \rightarrow \infty} \operatorname{Re} \mathfrak{a}(t_n, u(t_n) - Pu(t_n), u(t_n) - Pu(t_n)) \\ & \leq \limsup_{n \rightarrow \infty} \operatorname{Re} \langle \mathcal{A}(t_n)u(t_n) - F(t_n, Pu(t_n)), u(t_n) - Pu(t_n) \rangle \\ & = \operatorname{Re} \langle \mathcal{A}(c)u(c) - F(c, Pu(c)), u(c) - Pu(c) \rangle, \end{aligned}$$

where we used that \mathfrak{a} is right-continuous and $v \mapsto \operatorname{Re} \mathfrak{a}(c, v, v) + \omega \|v\|_H^2$ is an equivalent norm on V for the first inequality and (6.4) with $t = t_n$ for the second inequality. This shows $P\tilde{V} \subset V$ and

$$\operatorname{Re} \mathfrak{a}(t, Pv, v - Pv) \geq \operatorname{Re} \langle F(t, Pv), v - Pv \rangle \quad (t \in [a, b], v \in \tilde{V}).$$

Finally let $v \in V$ and $(v_n)_{n \in \mathbb{N}} \subset \tilde{V}$, $v_n \rightarrow v$ in V . With a similar argument as above (where we replace the role of $u(t_n)$ by v_n and $u(c)$ by v) we obtain the assertion of the corollary. \square

7 An illustrating example

In this section we show by an example how the invariance criterion Theorem 3.1 can be applied. We consider an elliptic operator of second order with time-dependent coefficients. Let $\Omega \subset \mathbb{R}^d$ be open and bounded, let H be the real Hilbert space $L^2(\Omega)$ and let V be $H^1(\Omega)$ if we consider Neumann boundary conditions or $H_0^1(\Omega)$ for Dirichlet boundary conditions. Let $T > 0$. We assume that $a_{jk}: [0, T] \times \Omega \rightarrow [0, \infty)$, $j, k \in \{1, \dots, d\}$ are measurable, bounded by some constant $M \geq 0$ and that there exists some constant $\alpha > 0$ such that

$$\sum_{j,k=1}^d a_{jk}(t, x) \xi_j \xi_k \geq \alpha |\xi|^2 \quad (\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d)$$

for a.e. $(t, x) \in [0, T] \times \Omega$. We define $\mathbf{a}: [0, T] \times V \times V \rightarrow \mathbb{R}$ by

$$\mathbf{a}(t, v, w) = \sum_{j,k=1}^d \int_{\Omega} a_{jk}(t, x) \partial_j v \partial_k w \, dx.$$

Then $\mathbf{a} \in ([0, T]; V, H)$, where (2.1) holds with the same constant M and (2.2) holds with the same α and $\omega = \alpha$.

Let $F: \mathbb{R} \rightarrow \mathbb{R}$ be *locally Lipschitz*, i.e. F is Lipschitz continuous on bounded subsets of \mathbb{R} and suppose that $F(0) = F(1) = 0$.

Proposition 7.1. *For every $u_0 \in H$ with $u_0 \in [0, 1]$ a.e., there exists a unique $u \in MR([0, T])$ such that $u(t) \in [0, 1]$ a.e. for every $t \in [0, T]$ and*

$$u'(t) + \mathcal{A}u(t) = F(u(t)) \quad (\text{a.e. } t \in [0, T]), \quad u(0) = u_0. \quad (7.1)$$

Note that $x \mapsto F(u(t)(x)) \in H$ for every $t \in [0, T]$ since Ω is bounded and $MR([0, T]) \hookrightarrow C([0, T]; H)$. Thus (7.1) makes sense. For example, if $F(x) = x(1 - x)$, then (7.1) is a diffusion equation with logistic growth.

Proof. Let $\mathcal{C} := \{g \in H : g \in [0, 1] \text{ a.e.}\}$. Then \mathcal{C} is a closed and convex subset of H and the orthogonal projection $P: H \rightarrow H$ onto \mathcal{C} is given by $Pg(x) = \max\{\min\{g(x), 1\}, 0\} = \min\{\max\{g(x), 0\}, 1\}$. Moreover, $g - Pg = (g - 1)_+ - (-g)_+$. Thus $(F(Pg) | g - Pg)_H = 0$. Let $v \in V$, then $\nabla Pv = \nabla v \mathbf{1}_{\{0 \leq v \leq 1\}}$ and $\nabla(v - Pv) = \nabla v \mathbf{1}_{\{v < 0\} \cup \{v > 1\}}$. Thus $\mathbf{a}(t, Pv, v - Pv) = 0$ for every $t \in [0, T]$. Finally, Proposition 5.1 yields a unique solution $u \in MR([0, T])$ of

$$u'(t) + \mathcal{A}u(t) = F(Pu(t)) \quad (\text{a.e. } t \in [0, T]), \quad u(0) = u_0,$$

which is in \mathcal{C} for every $t \in [0, T]$ by Corollary 3.3. Thus $Pu(t) = u(t)$ for every $t \in [0, T]$ and hence u is our desired unique solution. \square

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